Representable Functors

Let $\mathcal{C}$ be a locally small category.

A representable functor $\mathcal{C} \longrightarrow \mathrm{Set}$ is a functor of the form $\mathrm{Hom}_{\mathcal{C}}(C,-)$ for some object C.

$$\mathcal{C} \xrightarrow{\mathrm{Hom}(C,-)} \mathrm{Set}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] & \mathrm{Hom}(C,A) \arrow[d, "\mathrm{Hom}(C,f)"'] \ni g \arrow[d, mapsto] \\
B & \mathrm{Hom}(C,B) \ni f \circ g
\end{tikzcd}$$

The contravariant Yoneda embedding:

$$y^{-} : \mathcal{C}^{op} \longrightarrow \mathrm{Set}^{\mathcal{C}}$$

$$\begin{tikzcd}
C \arrow[d, "h"'] & y^C = \mathrm{Hom}(C,-) & \mathrm{Hom}(C,A) \ni g \arrow[d, "(y^h)_A"'] \\
D & y^D = \mathrm{Hom}(D,-) \arrow[u, "y^h"] & \mathrm{Hom}(D,A) \ni g \circ h
\end{tikzcd}$$

Exercise: Prove naturality of y^h!

The covariant Yoneda embedding:

$$y_{-} : \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$$

$$\begin{tikzcd}
C \arrow[d, "h"'] & y_C = \mathrm{Hom}(-,C) \arrow[d, "y_h"'] & \mathrm{Hom}(A,C) \ni f \arrow[d, "(y_h)_A"'] \\
D & y_D = \mathrm{Hom}(-,D) & \mathrm{Hom}(A,D) \ni h \circ f
\end{tikzcd}$$

We will prove that

y_{-} is an embedding, ie

- It is injective on objects
- It is full and faithful:
  bijective on hom-sets.

So y_{-} makes $\mathcal{C}$ a subcategory of $\mathrm{Set}^{\mathcal{C}^{op}}$

presheaves on $\mathcal{C}$

$\mathrm{Set}^{\mathcal{C}^{op}}$ is a topos: a mathematical universe with its own internal logic, which allows many important constructions.

Yoneda Lemma

For every object C of $\mathcal{C}$

and functor (presheaf) $F : \mathcal{C}^{op} \longrightarrow Set$,

$$\mathrm{Nat}(y_C, F) \cong FC$$

$\uparrow$ natural transformations

This isomorphism is natural in C and F.

Proof

($\Rightarrow$) Given $\alpha \in \mathrm{Nat}(y_C, F)$, we construct $x_\alpha \in FC$

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "y_C = Hom(-{,}C)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

$$\begin{tikzcd} & y_C C = Hom(C,C) \arrow[d, "\alpha_C"'] & \ni & id_C \arrow[d, mapsto] \\ C & FC & \ni & \alpha_C(id_C) \end{tikzcd}$$

Define $\cancel{\alpha}\ x_\alpha = \alpha_C(id_C)$

($\Leftarrow$) Given $x \in FC$, we construct $\alpha_x \in \mathrm{Nat}(y_C, F)$

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "y_C"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha_x"] & Set \end{tikzcd}$$

$$\begin{tikzcd} & y_C(X) = Hom(X,C) \arrow[d, "(\alpha_x)_X"'] & \ni & f \arrow[d, mapsto] \\ X & FX & \ni & Ff(x) \end{tikzcd}$$

Define $(\alpha_x)_X = \lambda f.\ Ff(x)$

Exercise: Prove naturality of the isomorphism in C and F. ∎

Example

Grph can be seen as the sheaf category $Set^{\mathcal{C}^{op}}$ with $\mathcal{C}$ described by the diagram

$$\begin{tikzcd} v \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ e \end{tikzcd}$$

(identities implicit)

Exercise:

Verify that y_v represents the graph with a single vertex and no edges and y_e represents the graph with two distinct ~~edges~~ vertices and a single edge between them.

Remember that the simplex category Δ looks like this:

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r, shift left=2] \arrow[r, shift right=2] & 2 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & 3 \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

(only the generating face and degeneracy morphisms are shown)

An object of $Set^{\Delta^{op}}$ is calle a simplicial set

Intuitively, a simplicial set is a geometric object constructed with simplices, ie n-dimensional triangles.

If $X : \Delta^{op} \longrightarrow Set$

think of X_n as the set of simplices of dimension $n-1$:

X_1 are points

X_2 are lines

X_3 are triangles

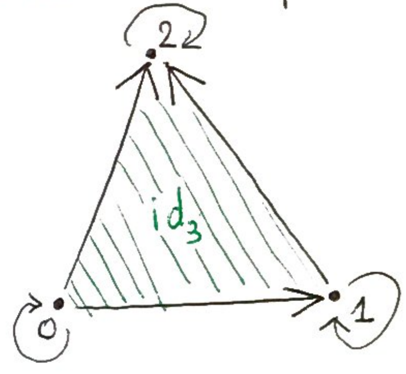
X_4 are tetrahedra ...

The images of the face maps tell us how the simplices are glued together.

The Yoneda embedding gives the standard simplices.

$Y_3 : \Delta^{op} \longrightarrow Set$

standard 2-simplex:



$Y_3 0 = Hom(0,3) \cong \mathbb{1}$

$Y_3 1 = Hom(1,3) \cong \{0,1,2\}$

$Y_3 2 = Hom(2,3) \cong$ {three lines + three degenerate lines}

$Y_3{}^3 = Hom(3,3) = \{ id_3$ + 6 line-degenerate triangles + 3 point-degenerate triangles$\}$

the actual simplex

$Y_3 4, \ldots$ contain just degenerate simplices.