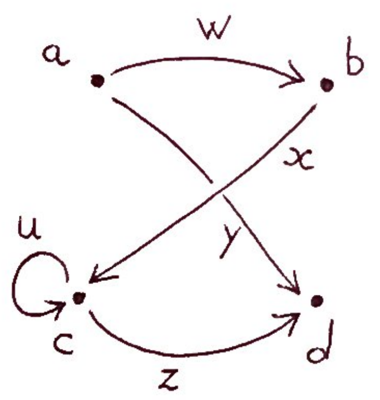
The Category of Graphs

A (directed) graph consists of:

- A set V of vertices
- A set E of edges
- Every edge has a source vertix and a target vertix

$\mathcal{G} = \langle V, E, s, t \rangle \qquad s, t : E \longrightarrow V$

Example:



$V = \{a, b, c, d\}$

$E = \{u, w, x, y, z\}$

$s(w) = a \qquad t(w) = b$

$s(u) = c \qquad t(u) = c$

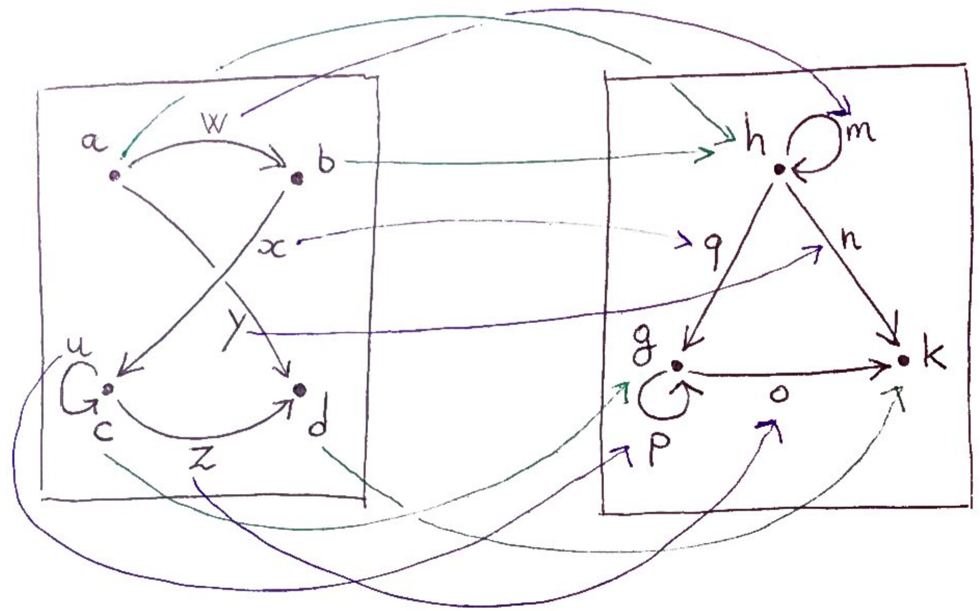
$\vdots \qquad \vdots$

What is a "function" between graphs?

As for any mathematical/algebraic structure, a function should:

- Map elements of the domain to elements of the codomain;
- Preserve the "relations" between elements.

$\mathcal{G}_1 \xrightarrow{\quad f \quad} \mathcal{G}_2$



We call such a function preserving relations

A __morphism__ between graphs:

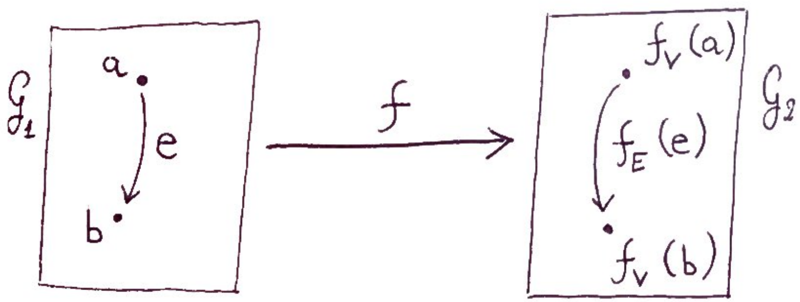
$$\mathcal{G}_1 \xrightarrow{\;f\;} \mathcal{G}_2$$

$$\mathcal{G}_1 = \langle V_1, E_1, s_1, t_1 \rangle \qquad \mathcal{G}_2 = \langle V_2, E_2, s_2, t_2 \rangle$$

$$f = \langle f_V, f_E \rangle \quad \text{with} \quad f_V : V_1 \longrightarrow V_2, \qquad f_E : E_1 \longrightarrow E_2$$

such that

$$\left.\begin{aligned} f_V(s_1(e)) &= s_2(f_E(e)) \\ f_V(t_1(e)) &= t_2(f_E(e)) \end{aligned}\right\} \text{ f preserves source and target}$$



__Identity__ morphisms:

$$id_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G} \qquad id_{\mathcal{G}}(v) = v, \quad id_{\mathcal{G}}(e) = e$$

__Composition__ of morphisms:

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathcal{G}_2 \arrow[r, "g"] & \mathcal{G}_3 \end{tikzcd}$$

$$(g \circ f)_V(v) = g_V(f_V(v))$$

$$(g \circ f)_E(e) = g_E(f_E(e))$$

Exercise 1:

Verify that if f and g are graph morphisms, so is $g \circ f$;
i.e. if f and g preserve source and target, then also $g \circ f$ preserves source and target.

Observation (Exercise 2)

Identity and composition satisfy the following equalities:

$$f \circ id_{\mathcal{G}_1} = f \qquad id_{\mathcal{G}_2} \circ f = f$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathcal{G}_2 \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & \mathcal{G}_3 \arrow[r, "h"] & \mathcal{G}_4
\end{tikzcd}$$

Note:

Simple sets also have identity functions and a composition operation satisfying the same equalities.

Other mathematical structures have their own definition of morphism.

There are always identity morphisms and a composition operation satisfying the equations.

This leads to the abstract notion of <u>category</u>.

A category is a system of mathematical structures of a certain kind, together with morphisms between them.